

# PRODUCT OF VOLTERRA TYPE INTEGRAL AND COMPOSITION OPERATORS ON WEIGHTED FOCK SPACES

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**ABSTRACT.** We characterize the bounded, compact, and Schatten class product of Volterra type integral and composition operators acting between weighted Fock spaces. Our results are expressed in terms of certain Berezin type integral transforms on the complex plane  $\mathbb{C}$ . We also estimate the norms and essential norms of these operators in terms of the integral transforms. All our results are valid for weighted composition operators when acting between the class of weighted Fock spaces considered.

## 1. INTRODUCTION

Given a space of functions  $\mathcal{H}$  of holomorphic functions on  $\mathbb{C}$ , we define the Volterra type integral operator on  $\mathcal{H}$  induced by a holomorphic symbol  $g$  by

$$V_g f(z) = \int_0^z f(w)g'(w)dw.$$

Questions about boundedness, compactness, and other operator theoretic properties of  $V_g$  expressed in terms of function theoretic conditions on  $g$  have been a subject of high interest since introduced by Pommerenke [15] in 1997. The operator  $V_g$  has in particular attracted a lot of interest following the works of Aleman and Siskakis [2, 3] on Hardy and Bergmann spaces. For more information, we refer to the surveys in [1, 18] and the references therein. The Volterra type integral operator  $V_g$  has an interesting relation with the multiplication operator  $Mg(f) = gf$  by  $M_g(f) = f(0)g(0) + V_g(f) + I_g(f)$ , where  $I_g$  is the Volterra companion integral operator given by

$$I_g f(z) = \int_0^z f'(w)g(w)dw. \quad (1.1)$$

Let  $\psi$  be entire function and  $C_\psi f = f(\psi)$  be the induced composition operator on space of analytic functions on  $\mathbb{C}$ . We define the product of Volterra type integral and composition operators induced by the pair of symbols  $(g, \psi)$  by

$$V_{(g,\psi)} f(z) = \int_0^z f(\psi(w))g'(w)dw. \quad (1.2)$$

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If  $\psi(z) = z$ , then these operators are just the usual Volterra type integral operators  $V_g$ . As will be seen latter the study of  $V_{(g,\psi)}$  reduce to studying the composition operator  $C_\psi$  when  $|g'(z)/(1 + |z|)|$  behaves like a constant for all  $z$ . Several authors have studied operators of this kind [11, 13, 17, 22, 24]. Ž. Čučković and R. Zhao [6, 7] characterized the bounded and compact weighted composition operators between different weighted Bergman spaces and different Hardy spaces in terms of the generalized Berezin transform. Similar results were also obtained in [20] for the same operator acting on the classical Fock space  $F_1^2$ .

In this paper, we present analogous results for product of Volterra type integral and composition operator  $V_{(g,\psi)}$  when it acts between different weighted Fock spaces. By modifying all the results stated for  $V_{(g,\psi)}$ , one could also obtain similar results for the operators

$$C_{(\psi,g)}f(z) := \int_0^z f(\psi(w))(g(\psi(w)))' dm(w).$$

We recall that for  $0 < p < \infty$  and  $\alpha > 0$ , the Fock space  $F_\alpha^p$  consists of entire functions  $f$  for which

$$\|f\|_{(p,\alpha)}^p = \frac{p\alpha}{2\pi} \int_{\mathbb{C}} |f(z)|^p e^{-\frac{p\alpha}{2}|z|^2} dm(z) < \infty$$

where  $dm$  is the Lebesgue measure. In particular,  $F_\alpha^2$  is a reproducing kernel Hilbert space with kernel and normalized kernel functions respectively  $K_{(w,\alpha)}(z) = e^{\alpha\bar{w}z}$  and  $k_{(w,\alpha)}(z) = e^{-\alpha|w|^2/2 + \alpha\bar{w}z}$ .

Our results will be expressed in terms of the Berezin type integral transform

$$B_{(\psi,\alpha)}(|g|^p)(w) = \int_{\mathbb{C}} e^{\frac{p\alpha}{2}(2\Re\langle\psi(z), w\rangle - |z|^2 - |w|^2)} \frac{|g'(z)|^p}{(1 + |z|)^p} dm(z),$$

where  $\langle \cdot, \cdot \rangle$  is the standard inner product in the complex plan  $\mathbb{C}$ .

**1.1. bounded and compact  $V_{(g,\psi)}$ .** We now state our first main result.

**Theorem 1.** *Let  $0 < p \leq q < \infty$  and  $\psi$  be an entire function. Then  $V_{(g,\psi)} : F_\alpha^p \rightarrow F_\alpha^q$  is*

(i) *bounded if and only if  $B_{(\psi,\alpha)}(|g|^q) \in L^\infty(\mathbb{C}, dm)$ . Moreover<sup>1</sup>*

$$\|V_{(g,\psi)}\| \simeq \left( \sup_{w \in \mathbb{C}} B_{(\psi,\alpha)}(|g|^q)(w) \right)^{1/q}. \quad (1.3)$$

(ii) *compact if and only if  $\lim_{|z| \rightarrow \infty} B_{(\psi,\alpha)}(|g|^q)(z) = 0$ .*

The conditions both in (i) and (ii) are independent of the exponent  $p$  apart from the fact that  $p$  should not be exceeding  $q$ . It means that if there exists a  $p > 0$  for which  $V_{(g,\psi)}$  is bounded (compact) from  $F_\alpha^p$  to  $F_\alpha^q$ , then it is also bounded (compact) for every other  $p \leq q$ .

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<sup>1</sup>The notation  $U(z) \lesssim V(z)$  (or equivalently  $V(z) \gtrsim U(z)$ ) means that there is a constant  $C$  such that  $U(z) \leq CV(z)$  holds for all  $z$  in the set in question, which may be a Hilbert space or a set of complex numbers. We write  $U(z) \simeq V(z)$  if both  $U(z) \lesssim V(z)$  and  $V(z) \lesssim U(z)$ .

It is often difficult to determine whether a concrete operator on a function space possesses properties such as boundedness, compactness or Schatten class membership. For reproducing kernel Hilbert spaces, a fruitful strategy has been to test whether the operator theoretic properties could be determined by its action on the kernel functions alone. In general there is no reason why this should hold but many important results are known to be interpreted as examples of this property which we call the reproducing kernel thesis property. Our results above present another example of the thesis property. The boundedness and compactness of  $V_{(g,\psi)}$  are respectively equivalent to

$$\sup_{z \in \mathbb{C}} \|V_{(g,\psi)}k_{(z,\alpha)}\|_{(q,\alpha)} < \infty \text{ and } \lim_{|z| \rightarrow \infty} \|V_{(g,\psi)}k_{(z,\alpha)}\|_{(q,\alpha)} = 0.$$

A natural question is whether there exists an interplay between the two symbols  $g$  and  $\psi$  in inducing bounded and compact operators  $V_{(g,\psi)}$ . We first observe that if  $g' \neq 0$ , then by the classical Liouville's theorem the function  $g$  can not decay in any way. This forces that

$$B_{(\psi,\alpha)}(|g|^p)(w) = \int_{\mathbb{C}} e^{\frac{p\alpha}{2}(|\psi(z)|^2 - |z|^2 - |\psi(z)-w|^2)} \frac{|g'(z)|^p}{(1+|z|)^p} dm(z)$$

is bounded only when  $\psi(z) = az + b$  with  $|a| \leq 1$ . Moreover if  $|a| = 1$ , then  $b = 0$ , and compactness is achieved only when  $|a| < 1$ . Combining this with Proposition 3 in [4], we get the following<sup>2</sup>.

**Corollary 1.** *Let  $0 < p \leq q < \infty$ ,  $g' \neq 0$  and  $\psi$  be an entire function. Then if  $V_g \circ C_\psi = V_{(g,\psi)} : F_\alpha^p \rightarrow F_\alpha^q$  is*

- (i) *bounded, then  $C_\psi$  is bounded.*
- (ii) *compact, then  $C_\psi$  is compact.*

On the other hand a bounded  $V_{(g,\psi)}$  does not necessarily imply boundedness of the Volterra type integral operator  $V_g$ . This is because boundedness of the former allows  $g$  to be any entire function that grows more slowly than the exponential part of the integrand in  $B_{(\psi,\alpha)}(|g|^p)(w)$  while boundedness of the latter forces  $g$  to grow as a power function of at most degree 2, as seen below. By setting  $\psi(z) = z$  in the theorem, we immediately get the following result of Constantin [5].

**Corollary 2.** *Let  $0 < p \leq q < \infty$ . Then  $V_g : F_\alpha^p \rightarrow F_\alpha^q$  is*

- (i) *bounded if and only if  $g(z) = az^2 + bz + c$ ,  $a, b, c \in \mathbb{C}$ .*
- (ii) *compact if and only if  $g(z) = az + b$ .*

*Proof.* By Theorem 1, the sufficiency of the conditions both in (i) and (ii) are immediate. We shall sketch the necessity. If  $D(w, 1) = \{z \in \mathbb{C} : |z - w| < 1\}$ , then

$$B_{(\psi,\alpha)}(|g|^q)(w) \gtrsim \int_{D(w,1)} \frac{|g'(z)|^q}{(1+|z|)^q} dm(z) \gtrsim \frac{|g'(w)|^q}{(1+|w|)^q}, \quad (1.4)$$

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<sup>2</sup>The Fock type spaces in [4] are defined in a slightly different way than ours.

where the last inequality follows by subharmonicity. Assuming boundedness of  $V_{(g,\psi)}$ , (1.4) implies  $|g'(w)| \lesssim 1 + |w|$  for all  $w \in \mathbb{C}$ , from which the desired expression for  $g$  follows. On the other hand, if  $V_g$  is compact, then since  $k_{(w,\alpha)} \rightarrow 0$  uniformly on compact subsets of  $\mathbb{C}$  as  $|w| \rightarrow \infty$  we see from relation (1.4) that

$$\frac{|g'(w)|}{1 + |w|} \rightarrow 0, \text{ as } |w| \rightarrow \infty.$$

This can happen only when  $g$  is a polynomial of degree at most 1. □

Interestingly, many more  $g$ 's are admissible than those in the previous corollary if we scale  $\psi$  as  $\psi(z) = \beta z$  with  $|\beta| < 1$ . More precisely, we get the following whose proof is just immediate from the theorem.

**Corollary 3.** *Let  $\psi(z) = \beta z$  with  $|\beta| < 1$  and  $0 < p \leq q < \infty$ . Then  $V_{(g,\psi)} : F_\alpha^p \rightarrow F_\alpha^q$  is bounded for any  $g$  such that*

$$|g(z)| \lesssim e^{\frac{\alpha\gamma}{2}|z|^2}$$

for all  $z$  in  $\mathbb{C}$  and any  $\gamma$  satisfying  $\gamma + |\beta|^2 < 1$ .

Theorem 1 and all the subsequent results are valid for the weighted composition operator  $(uC_\psi)f(z) = u(z).f(\psi(z))$  between the Fock spaces as described here where  $u$  is an entire function on  $\mathbb{C}$ . We only have to replace the weight  $|g'(z)|/(1 + |z|)$  by  $|u(z)|$  to get the corresponding results. For  $p = q = 2$  and  $\alpha = 1$ , the bounded and compact composition operators were described in [20] apart from missing the fact that  $\psi$  can be nothing but linears.

For the case where we map larger weighted Fock spaces into smaller ones, we get the following stronger conditions as one would expect.

**Theorem 2.** *Let  $0 < q < p < \infty$  and  $\psi$  be an entire function. Then the following are equivalent*

- (i)  $V_{(g,\psi)} : F_\alpha^p \rightarrow F_\alpha^q$  is bounded.
- (ii)  $V_{(g,\psi)} : F_\alpha^p \rightarrow F_\alpha^q$  is compact.
- (iii)  $B_{(\psi,\alpha)}(|g|^q) \in L^{\frac{p}{p-q}}(\mathbb{C}, dm)$ . Moreover,

$$\|V_{(g,\psi)}\| \simeq \left( \int_{\mathbb{C}} B_{(\psi,\alpha)}^{p/(p-q)}(|g|^q)(w) dm(w) \right)^{(p-q)/p} \quad (1.5)$$

It is interesting to note that unlike condition (iii) of Theorem 1 where we map smaller spaces into bigger ones, condition (iii) above is expressed in terms of both exponents  $p$  and  $q$ . When  $\psi(z) = z$ , then the theorem simplifies to saying that  $V_g$  (for non constant  $g$ ) is bounded or compact if and only if  $g'$  is a constant  $q > 2p/(p+2)$  and  $g' = 0$  for

$q < 2p/(p+2)$ . This is because by subharmonicity,

$$\begin{aligned} \int_{\mathbb{C}} B_{(\psi, \alpha)}^{\frac{p}{p-q}}(|g|^q)(w) dm(w) &\gtrsim \int_{\mathbb{C}} \left( \int_{D(w, 1)} \left| \frac{k_{(w, \alpha)}(z) g'(z)}{(1+|z|)} \right|^q e^{-\frac{q\alpha}{2}|z|^2} dm(z) \right)^{\frac{p}{p-q}} dm(w) \\ &\geq \int_{\mathbb{C}} \left| \frac{g'(w)}{1+|w|} \right|^{1/q} dm(w) \end{aligned} \quad (1.6)$$

from which the desired restrictions on  $g$ ,  $p$  and  $q$  follow once we assume that the left-hand side of (1.6) is finite.

Observe that by setting  $|g'(z)/(1+|z|)| \simeq 1$ , Theorem 2 characterizes the bounded and compact composition operators from  $F_{\alpha}^p$  to  $F_{\alpha}^q$  whenever  $p > q$ . This extends the result in [4] where similar conditions are given for compact and bounded  $C_{\psi} : F_{\alpha}^p \rightarrow F_{\alpha}^q$  whenever  $0 < p \leq q < \infty$ . Those conditions in [4] for the one variable setting could also be obtained easily from Theorem 1. The corresponding conditions in Theorem 2 can be easily simplified to give that  $C_{\psi} : F_{\alpha}^p \rightarrow F_{\alpha}^q$  is bounded (compact) if and only if  $\psi(z) = az + b$  with  $|a| < 1$ .

**1.2. Essential norm of  $V_{(g, \psi)}$ .** The essential norm  $\|T\|_e$  of a bounded operator  $T$  on a Banach space  $\mathcal{H}$  is defined as the distance from  $T$  to the space of compact operators on  $\mathcal{H}$ . We refer to [6, 7, 16, 20, 25] for estimation of such norms for different operators on Hardy space, Bergman space,  $L^p$  and some Fock spaces. We get the following estimate for  $V_{(g, \psi)}$ .

**Theorem 3.** *Let  $1 < p \leq q < \infty$  and  $\psi$  be an entire function. If  $V_{(g, \psi)} : F_{\alpha}^p \rightarrow F_{\alpha}^q$  is bounded, then*

$$\|V_{(g, \psi)}\|_e \simeq \left( \limsup_{|w| \rightarrow \infty} B_{(\psi, \alpha)}(|g|^q)(w) \right)^{1/q}. \quad (1.7)$$

For  $p > 1$ , the compactness condition in Theorem 1 could be easily drawn from this relation since the left-hand side expression (1.7) vanishes for compact  $V_{(g, \psi)}$ . In particular when  $\psi(z) = z$ , a simple computation along with Corollary 2 shows that

$$\|V_g\|_e \simeq \sqrt[q]{q}.$$

**1.3. Schatten Class  $V_{(g, \psi)}$ .** Let us now turn to the Schatten class membership of  $V_{(g, \psi)}$ . We recall that a positive operator  $T$  on  $F_{\alpha}^2$  belongs to the trace class if

$$\sum_{n=1}^{\infty} \langle T e_n, e_n \rangle < \infty$$

for some orthonormal basis  $(e_n)$  of  $F_{\alpha}^2$ . If  $0 < p < \infty$ , a bounded operator  $T$  on  $F_{\alpha}^2$  belongs to the Schatten class  $S_p$  if the positive operator  $(T^*T)^{p/2}$  is in the trace class. We denote the  $S_p$  norm of  $T$  by  $\|T\|_{S_p}$ .

**Proposition 1.** *Let  $\mathcal{H}$  be any Hilbert space and  $T$  be a bounded operator from  $F_\alpha^2$  to  $\mathcal{H}$ . (i) If  $p \geq 2$  and  $T \in S_p$ , then*

$$\int_{\mathbb{C}} \|Tk_{(z,\alpha)}\|_{\mathcal{H}}^p dm(z) < \infty. \quad (1.8)$$

(ii) If  $0 < p \leq 2$  and (1.8) holds, then  $T \in S_p$ .

It is shown in [9] that the converse to the two statements above fail to hold for Hankel operators on the Hardy space  $H^2$ . The interest is now whether the converse still hold for the product of Volterra type integral and composition operators under consideration. It turns out that this is indeed the case (see, Theorem 4). In particular,  $T$  belongs to the Hilbert–Schmidt class if and only if for any orthonormal basis  $(e_n)$  in  $\mathcal{H}$ ,

$$\begin{aligned} \|T\|_{S_2}^2 &= \sum_{n=1}^{\infty} \|T^*e_n\|_{(2,\alpha)}^2 \simeq \sum_{n=1}^{\infty} \int_{\mathbb{C}} T^*e_n(z) \overline{T^*e_n(z)} e^{-\alpha|z|^2} dm(z) \\ &= \int_{\mathbb{C}} \sum_{n=1}^{\infty} |\langle TK_{(z,\alpha)}, e_n \rangle|^2 e^{-\alpha|z|^2} dm(z) \\ &= \int_{\mathbb{C}} \|Tk_{(z,\alpha)}\|_{\mathcal{H}}^2 dm(z) < \infty. \end{aligned} \quad (1.9)$$

If  $T$  is any positive operator in the trace class of  $F_\alpha^2$ , then by the above

$$tr(T) = \|T^{1/2}\|_{S_2}^2 \simeq \int_{\mathbb{C}} \|T^{1/2}k_{(z,\alpha)}\|_{(2,\alpha)}^2 dm(z) = \int_{\mathbb{C}} \langle Tk_{(z,\alpha)}, k_{(z,\alpha)} \rangle dm(z).$$

We recall that  $T$  belongs to the Schatten class  $S_p$  if and only if  $(T^*T)^{p/2}$  belongs to the trace class. Thus

$$tr((T^*T)^{p/2}) \simeq \int_{\mathbb{C}} \langle (T^*T)^{p/2}k_{(z,\alpha)}, k_{(z,\alpha)} \rangle dm(z) \gtrsim \int_{\mathbb{C}} \|Tk_{(z,\alpha)}\|_{\mathcal{H}}^p dm(z),$$

for  $p \geq 2$  and the inequality is reversed for  $0 < p \leq 2$ . In particular when  $T = V_{(g,\psi)}$ , we have

$$\int_{\mathbb{C}} \|V_{(g,\psi)}k_{(z,\alpha)}\|_{(2,\alpha)}^p dm(z) \simeq \int_{\mathbb{C}} B_{(\psi,\alpha)}^{p/2}(|g|^2)(w) dm(w),$$

which gives the proofs of the necessity for  $p \geq 2$  and the sufficiency for  $0 < p \leq 2$  of our next theorem.

**Theorem 4.** *Let  $0 < p < \infty$  and  $\psi$  be an entire function. Then a bounded map  $V_{(g,\psi)} : F_\alpha^2 \rightarrow F_\alpha^2$  belongs to  $S_p$  if and only if  $B_{(\psi,\alpha)}(|g|^2) \in L^{p/2}(\mathbb{C}, dm)$ .*

In particular for the Volterra type integral operator  $V_g$ , we obtain the following.

**Corollary 4.** *Let  $V_g$  be a compact operator on  $F_\alpha^2$ . If  $0 < p \leq 2$ , then  $V_g$  belongs to  $S_p$  if and only if  $g$  is a constant function. On the other hand,  $V_g$  belongs to  $S_p$  for all  $p > 2$ .*

*Proof.* For  $p \geq 2$ , this result was also proved in [5]. Now we observe that it in fact follows immediately from Theorem 4. For (i), we observe that it suffices to show that there are no nontrivial Hilbert–Schmidt Volterra type integral operators. The rest will follow from the monotonicity property of the Schatten classes. To this end, if  $V_g$  is a compact operator, then by Corollary 2,  $g' = C$ , a constant. On the other hand by subharmonicity

$$\begin{aligned} \int_{\mathbb{C}} B_{(\psi, \alpha)}^{p/2}(|g|^2)(w) dm(w) &\simeq \int_{\mathbb{C}} \left( \int_{\mathbb{C}} \frac{C^2 e^{-|z-w|^2}}{(1+|z|)^2} dm(z) \right)^{p/2} dm(w) \\ &\gtrsim \int_{\mathbb{C}} \frac{C^p}{(1+|w|)^p} dm(w). \end{aligned}$$

Theorem 4 ensures that if  $V_g$  belongs to  $S_p$ , then the above integrals should converge. But for  $p = 2$ , this holds only when  $C = 0$ . The integral converges for all  $p > 2$ .  $\square$

By combining Theorem 4 with Funbini’s Theorem, it is also easily seen that  $V_{(g, \psi)}$  is a Hilbert–Schmidt operator if and only if

$$\int_{\mathbb{C}} \frac{|g'(z)|^2}{(1+|z|)^2} e^{\alpha|\psi(z)|^2 - \alpha|z|^2} dm(z) < \infty.$$

As remarked earlier, all our results are valid for weighted composition operator  $uC_\psi$ . For such operators, Theorem 4 can be simplified further.

**Corollary 5.** *Let  $0 < p < \infty$  and  $\psi$  and  $u$  be entire functions. Then a bounded map  $uC_\psi : F_\alpha^2 \rightarrow F_\alpha^2$  belongs to  $S_p$  if and only if  $\psi(z) = az + b$  and*

$$\int_{\mathbb{C}} |u(z)|^p e^{\frac{p\alpha}{2}(|a|^2 - 1)|z|^2 + 2\Re\langle az, b \rangle} dm(z) < \infty$$

for some  $a, b \in \mathbb{C}$  and  $|a| < 1$ .

*Proof.* By Theorem 4,  $uC_\psi \in S_p$  if and only if

$$\begin{aligned} \int_{\mathbb{C}} \left( \int_{\mathbb{C}} |u(z)|^2 |k_{(w, \alpha)}(\psi(z))|^2 e^{-\alpha|z|^2} dm(z) \right)^{p/2} dm(w) \\ \simeq \int_{\mathbb{C}} \|uC_\psi k_{(w, \alpha)}\|_{(2, \alpha)}^p dm(w) < \infty. \end{aligned}$$

On the other hand,  $uC_\psi \in S_p$  if and only if  $(uC_\psi)^* \in S_p$ , and  $\|uC_\psi\|_{S_p} = \|(uC_\psi)^*\|_{S_p}$ . From this it follows that

$$\int_{\mathbb{C}} \|uC_\psi k_{(w, \alpha)}\|_{(2, \alpha)}^p dm(w) = \int_{\mathbb{C}} \|(uC_\psi)^* k_{(w, \alpha)}\|_{(2, \alpha)}^p dm(w). \quad (1.10)$$

Since  $(uC_\psi)^* k_{(w, \alpha)} = \overline{u(w)} e^{-\alpha|w|^2/2} K_{(\psi(w), \alpha)}$ , we find

$$\|(uC_\psi)^* k_{(w, \alpha)}\|_{(2, \alpha)} = |u(w)| e^{\frac{\alpha}{2}(|\psi(w)|^2 - |w|^2)},$$

and plugging this into (1.10) gives that  $uC_\psi \in S_p$  if and only if

$$\int_{\mathbb{C}} |u(z)|^p e^{\frac{p\alpha}{2}(|\psi(z)|^2 - |z|^2)} dm(z) < \infty.$$

Compactness forces that  $\psi(z) = az + b$ ,  $|a| < 1$  and hence the desired result follows.

Note that the above argument can not be carried over in general to simplify Theorem 4. Combining Corollary 5 with Theorem 1 immediately gives the following known Schatten class membership criteria for the composition operator.

**Corollary 6.** *Let  $0 < p < \infty$  and  $\psi$  be an entire function. Then the following are equivalent for a bounded map  $C_\psi$ .*

- (i) *The map  $C_\psi : F_\alpha^2 \rightarrow F_\alpha^2$  is compact.*
- (ii) *The map  $C_\psi : F_\alpha^2 \rightarrow F_\alpha^2$  belongs to  $S_p$  for all  $p > 0$ .*
- (iii)  *$\psi(z) = az + b$ ,  $a, b \in \mathbb{C}$ , and  $|a| < 1$ .*

## 2. PROOF OF THE MAIN RESULTS

One of the main tools in proving our results is the following lemma.

**Lemma 1.** *Let  $f$  be an entire function and  $0 < p < \infty$ . Then*

$$\int_{\mathbb{C}} |f(z)|^p e^{-\frac{\alpha p}{2}|z|^2} dm(z) \simeq |f(0)|^p + \int_{\mathbb{C}} \frac{|f'(z)|^p}{(1+|z|)^p} e^{-\frac{\alpha p}{2}|z|^2} dm(z).$$

The lemma was proved by Constantin [5] and describes Fock spaces in terms of derivatives as analogous to the case of Bergman spaces in [14]. The following estimate will be needed frequently in our consideration later.

**Lemma 2.** *For each  $p > 0$ , let  $\mu_{(p,\alpha)}$  be positive pull back measure on  $\mathbb{C}$  defined by*

$$\mu_{(p,\alpha)}(E) = \frac{\alpha p}{2\pi} \int_{\psi^{-1}(E)} \frac{|g'(z)|^p}{(1+|z|)^p} e^{-\frac{\alpha p}{2}|z|^2} dm(z)$$

*for every Borel subset  $E$  of  $\mathbb{C}$ . Then*

$$\int_{D(w,1)} e^{\frac{\alpha p}{2}|z|^2} d\mu_{(p,\alpha)}(z) \lesssim e^{\frac{p\alpha}{2}} B_{(\psi,\alpha)}(|g|^p)(w).$$

*Proof.* For  $p = 2$ , the lemma was proved in [20]. A modification of that proof works for other  $p$ 's which we sketch it now. For each  $z \in D(w, 1)$ , observe that

$$|k_{(w,\alpha)}(z)|^p = |e^{\alpha \bar{w}z - \frac{\alpha}{2}|w|^2}|^p = e^{\frac{\alpha p}{2}(|z|^2 - |z-w|^2)} \geq e^{-\frac{\alpha p}{2} + \frac{p\alpha}{2}|z|^2}.$$

This implies

$$\begin{aligned} e^{-\frac{\alpha p}{2}} \int_{D(w,1)} e^{\frac{p\alpha}{2}|z|^2} d\mu_{(p,\alpha)}(z) &\leq \int_{D(w,1)} |k_{(w,\alpha)}(z)|^p d\mu_{(p,\alpha)}(z) \\ &\leq \int_{\mathbb{C}} |k_{(w,\alpha)}(z)|^p d\mu_{(p,\alpha)}(z). \end{aligned}$$



Invoking the definition of the measure  $\mu_{(p,\alpha)}$  and the integral transform  $B_{(\psi,\alpha)}(|g|^p)$  give

$$\begin{aligned} \int_{\mathbb{C}} |k_{(w,\alpha)}(z)|^p d\mu_{(p,\alpha)}(z) &\simeq \int_{\mathbb{C}} |k_{(w,\alpha)}(\psi(z))|^p \frac{|g'(z)|^p}{(1+|z|)^p} e^{-\frac{p\alpha}{2}|z|^2} dm(z) \\ &= B_{(\psi,\alpha)}(|g|^p)(w). \end{aligned}$$

*Proof of Theorem 1.* (i) Suppose that  $V_{(g,\psi)}$  is bounded. Then a simple computation shows that  $\|k_{w,\alpha}\|_{(p,\alpha)} = 1$  for all  $p > 0$ . Thus applying  $V_{(g,\psi)}$  on the normalized kernel functions along with Lemma 1 yields

$$1 \gtrsim \|V_{(g,\psi)}k_{(w,\alpha)}\|_{(q,\alpha)}^q \simeq B_{(\psi,\alpha)}(|g|^q)(w), \quad (2.1)$$

from which the necessity follows. To prove the sufficiency we extend the techniques used in [20]. By definition of the measure  $\mu_{(q,\alpha)}$ , Lemma 1, and Lemma 1 in [12]

$$\begin{aligned} \|V_{(g,\psi)}f\|_{(q,\alpha)}^q &\simeq \int_{\mathbb{C}} |f(z)|^q d\mu_{(q,\alpha)}(z) \\ &\lesssim \int_{\mathbb{C}} e^{\frac{q\alpha}{2}|z|^2} d\mu_{(q,\alpha)}(z) \int_{\mathbb{C}} \chi_{D(z,1)}(w) |f(w)|^q e^{-\frac{q\alpha}{2}|w|^2} dm(w) \end{aligned}$$

where  $\chi_{D(z,1)}$  is the characteristic function of  $D(z, 1)$ . By Lemma 2, Fubini's Theorem and the fact that  $\chi_{D(z,1)}(w) = \chi_{D(w,1)}(z)$ , for all  $z, w \in \mathbb{C}$ , we get

$$\begin{aligned} \|V_{(g,\psi)}f\|_{(q,\alpha)}^q &\lesssim \int_{\mathbb{C}} e^{\frac{q\alpha}{2}|z|^2} d\mu_{(q,\alpha)}(z) \int_{\mathbb{C}} \chi_{D(z,1)}(w) |f(w)|^q e^{-\frac{q\alpha}{2}|w|^2} dm(w) \\ &= \int_{\mathbb{C}} |f(w)|^q e^{-\frac{q\alpha}{2}|w|^2} dm(w) \left( \int_{\mathbb{C}} \chi_{D(w,1)}(z) e^{\frac{q\alpha}{2}|z|^2} d\mu_{(q,\alpha)}(z) \right) dm(w) \\ &\lesssim \int_{\mathbb{C}} |f(w)|^q e^{-\frac{q\alpha}{2}|w|^2} B_{(\psi,\alpha)}(|g|^q)(w) dm(w) \end{aligned} \quad (2.2)$$

$$\begin{aligned} &\lesssim \sup_{w \in \mathbb{C}} B_{(\psi,\alpha)}(|g|^q)(w) \|f\|_{(q,\alpha)}^q \\ &\leq \sup_{w \in \mathbb{C}} B_{(\psi,\alpha)}(|g|^q)(w) \|f\|_{(p,\alpha)}^q \end{aligned} \quad (2.3)$$

where the last inequality follows by the inclusion  $F_{\alpha}^p \subseteq F_{\alpha}^q$ . From (2.1) and (2.3), we deduce that (1.3) holds.

For (ii), observe that  $k_{(w,\alpha)}(z) = e^{\alpha\bar{w}z - \frac{\alpha}{2}|w|^2} \rightarrow 0$  uniformly on compact subsets of  $\mathbb{C}$  as  $|w| \rightarrow \infty$ . It follows that

$$0 = \lim_{|w| \rightarrow \infty} \|V_{(g,\psi)}k_{(w,\alpha)}\|_{(q,\alpha)}^q \simeq \lim_{|w| \rightarrow \infty} B_{(\psi,\alpha)}(|g|^q)(w)$$

from which the necessity follows again. So we remain to show the sufficiency of the condition. To this end, we let  $f_n$  be a sequence of entire functions such that  $\sup_n \|f_n\|_{(p,\alpha)} < \infty$  and  $f_n \rightarrow 0$  uniformly on compact subset of  $\mathbb{C}$  as  $n \rightarrow \infty$ . Then following the same line of argument as in the proof of the sufficiency of Theorem 1, we obtain

$$\|V_{(g,\psi)}(f_n)\|_{(q,\alpha)}^q \lesssim \int_{\mathbb{C}} |f_n(w)|^q e^{-\frac{q\alpha}{2}|w|^2} B_{(\psi,\alpha)}(|g|^q)(w) dm(w) = I_n$$

Then for a fixed  $r > 0$ , we split  $I_n$  as

$$\begin{aligned} I_n &= \left( \int_{|w| \leq r} + \int_{|w| > r} \right) |f_n(w)|^q e^{-\frac{q\alpha}{2}|w|^2} B_{(\psi, \alpha)}(|g|^q)(w) dm(w) \\ &= I_{n1} + I_{n2} \end{aligned} \quad (2.4)$$

and estimate each piece independently. We first estimate  $I_{n1}$ .

$$\begin{aligned} \limsup_{n \rightarrow \infty} I_{n1} &= \limsup_{n \rightarrow \infty} \int_{|w| \leq r} |f_n(w)|^q e^{-\frac{q\alpha}{2}|w|^2} B_{(\psi, \alpha)}(|g|^q)(w) dm(w) \\ &\leq \limsup_{n \rightarrow \infty} \sup_{|w| \leq r} |f_n(w)|^q \int_{|w| \leq r} e^{-\frac{q\alpha}{2}|w|^2} B_{(\psi, \alpha)}(|g|^q)(w) dm(w) \\ &\lesssim \limsup_{n \rightarrow \infty} \sup_{|w| \leq r} |f_n(w)|^q \rightarrow 0, \text{ as } n \rightarrow \infty, \end{aligned}$$

since  $\sup_{w \in \mathbb{C}} B_{(\psi, \alpha)}(|g|^q)(w) < \infty$ . We need to make a similar conclusion for the second piece of integral

$$\begin{aligned} \limsup_{n \rightarrow \infty} I_{n2} &= \limsup_{n \rightarrow \infty} \int_{|w| > r} |f_n(w)|^q e^{-\frac{q\alpha}{2}|w|^2} B_{(\psi, \alpha)}(|g|^q)(w) dm(w) \\ &\leq \sup_{|w| > r} B_{(\psi, \alpha)}(|g|^q)(w) \limsup_{n \rightarrow \infty} \|f_n\|_{(q, \alpha)}^q \\ &\leq \sup_{|w| > r} B_{(\psi, \alpha)}(|g|^q)(w) \limsup_{n \rightarrow \infty} \|f_n\|_{(p, \alpha)}^q \end{aligned}$$

Since  $\sup_n \|f_n\|_{(p, \alpha)} < \infty$ , we see that the last expression in the right hand side above converges to zero when  $r \rightarrow \infty$ , and hence  $V_{(g, \psi)}(f_n) \rightarrow 0$  in  $F_\alpha^q$  as  $n \rightarrow \infty$ .

*Proof of Theorem 2.* Since (ii) obviously implies (i), we shall show that (iii)  $\Rightarrow$  (ii) and (i)  $\Rightarrow$  (iii). We first assume that  $B_{(\psi, \alpha)}(|g|^q) \in L^{p/(p-q)}(\mathbb{C}, dm)$ , and show that  $V_{(g, \psi)}$  is compact. Let  $f_n$  be a sequence of functions such that  $\sup_n \|f_n\|_{(p, \alpha)} < \infty$  and  $f_n$  converges to zero uniformly on compact subsets of  $\mathbb{C}$ . Then we proceed as in the proof of Theorem 1 until we get equation (2.4). We only need to modify our arguments in estimating the two piece of integrals  $I_{n1}$  and  $I_{n2}$ . Since  $f_n \rightarrow 0$  uniformly on compact subsets of  $\mathbb{C}$

$$\begin{aligned} \limsup_{n \rightarrow \infty} I_{n1} &= \limsup_{n \rightarrow \infty} \int_{|w| \leq r} |f_n(w)|^q e^{-\frac{q\alpha}{2}|w|^2} B_{(\psi, \alpha)}(|g|^q)(w) dm(w) \\ &\leq \limsup_{n \rightarrow \infty} \sup_{|w| \leq r} |f_n(w)|^q \int_{|w| \leq r} e^{-\frac{q\alpha}{2}|w|^2} B_{(\psi, \alpha)}(|g|^q)(w) dm(w) \\ &\lesssim \limsup_{n \rightarrow \infty} \sup_{|w| \leq r} |f_n(w)|^q \rightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned}$$

The last integral above converges because

$$\int_{|w| \leq r} e^{-\frac{q\alpha}{2}|w|^2} B_{(\psi, \alpha)}(|g|^q)(w) dm(w) \lesssim \left( \int_{|w| \leq r} B_{(\psi, \alpha)}^s(|g|^q)(w) dm(w) \right)^{\frac{1}{s}}$$

by Hölder's inequality where we set  $s = p/(p - q)$  for brevity. Again by Hölder's inequality we obtain,

$$\begin{aligned} \limsup_{n \rightarrow \infty} I_{n2} &= \limsup_{n \rightarrow \infty} \int_{|w| > r} |f_n(w)|^q e^{-\frac{q}{2}|w|^2} B_{(\psi, \alpha)}(|g|^q)(w) dm(w) \\ &\lesssim \left( \int_{|w| > r} B_{(\psi, \alpha)}^s(|g|^q)(w) dm(w) \right)^{\frac{1}{s}} \limsup_{n \rightarrow \infty} \|f_n\|_{(p, \alpha)}^q. \end{aligned}$$

Since  $\sup_n \|f_n\|_{(p, \alpha)} < \infty$ , we let  $r \rightarrow \infty$  in the above relation and with (2.5) we conclude that  $V_{(g, \psi)} \rightarrow 0$  as  $n \rightarrow \infty$ . Thus  $V_{(g, \psi)}$  is compact. Obviously, (i) follows from (ii). Thus our proof will be complete once we show that (iii) follows from (i). To this end, we observe that  $V_{(g, \psi)}$  is bounded if and only if

$$\begin{aligned} \int_{\mathbb{C}} |V_{(g, \psi)} f(z)|^q e^{-\frac{q\alpha}{2}|z|^2} dm(z) &\simeq \int_{\mathbb{C}} |f(z)|^q d\mu_{(q, \alpha)}(z) \\ &= \int_{\mathbb{C}} |f(z)|^q e^{-\frac{q\alpha}{2}|z|^2} d\lambda_{(q, \alpha)}(z) \\ &\lesssim \|f\|_{(p, \alpha)}^q \end{aligned} \quad (2.5)$$

where  $d\lambda_{(q, \alpha)}(z) = e^{\frac{q\alpha}{2}|z|^2} d\mu_{(q, \alpha)}(z)$ . The inequality in (2.5) means that  $\lambda_{(q, \alpha)}$  is a  $(p, q)$  Fock–Carleson measure. By Theorem 3.3 in [10], this holds if and only if

$$\widetilde{\lambda_{(q, \alpha)}}(w) = \int_{\mathbb{C}} |k_{(w, \alpha)}(z)|^q e^{-\frac{q\alpha}{2}|z|^2} d\lambda_{(q, \alpha)}(z) \in L^{p/(p-q)}(\mathbb{C}, dm). \quad (2.6)$$

Substituting back  $d\lambda_{(q, \alpha)}$  and  $d\mu_{(q, \alpha)}$  in terms of  $dm$ , we obtain

$$\begin{aligned} \widetilde{\lambda_{(q, \alpha)}}(w) &= \int_{\mathbb{C}} |k_{(w, \alpha)}(z)|^q e^{-\frac{q\alpha}{2}|z|^2} d\lambda_{(q, \alpha)}(z) = \int_{\mathbb{C}} |k_{(w, \alpha)}(z)|^q d\mu_{(q, \alpha)}(z) \\ &\simeq \int_{\mathbb{C}} |k_{(w, \alpha)}(z)|^q \left| \frac{g'(z)}{1 + |z|} \right|^q e^{-\frac{q\alpha}{2}|z|^2} dm(z) = B_{(\psi, \alpha)}(|g|^q)(w) \end{aligned}$$

We remain to prove the norm estimate in (1.5). But this can be easily seen as follows. Since  $\lambda_{(q, \alpha)}$  is an  $(p, q)$  Fock–Carleson measure, the series of norm estimates in Theorem 3.3 in [10] yields

$$\|V_{g, \psi}\| \simeq \left( \|\widetilde{\lambda_{(q, \alpha)}}\|_{L^{p/(p-q)}(\mathbb{C}, dm)} \right)^{1/q} \simeq \left( \|B_{(\psi, \alpha)}(|g|^q)\|_{L^{p/(p-q)}(\mathbb{C}, dm)} \right)^{1/q}$$

which completes the proof of the theorem.

Theorem 3 follows from application of Lemmas 1-2, Theorem 1 and appropriate combination of arguments used to prove similar results in [7, 19, 20, 21]. Recall that each entire function  $f$  can be expressed as  $f(z) = \sum_{k=0}^{\infty} p_k(z)$  where the  $p_k$ 's are polynomials of degree  $k$ . We consider a sequence of operators  $R_n$  defined by

$$(R_n f)(z) = \sum_{k=n}^{\infty} p_k(z).$$

It was proved in [8, 19] that  $\lim_{n \rightarrow \infty} \|R_n f\|_{(p, \alpha)} = 0$  for each  $f$  in  $F_\alpha^p$ , and hence  $\sup_n \|R_n\| < \infty$ . We need the following more lemma in proving the theorem.

**Lemma 3.** *Let  $1 < p \leq q < \infty$  and  $\psi$  be an entire function. If  $V_{(g, \psi)} : F_\alpha^p \rightarrow F_\alpha^q$  is bounded, then*

$$\|V_{(g, \psi)}\|_e \leq \liminf_{n \rightarrow \infty} \|V_{(g, \psi)} R_n\|_{(q, \alpha)}.$$

The proof of the lemma is similar to the proof of Lemma 2 in [21], and we omit it.

*Proof of Theorem 3.* We first prove the lower estimate in the theorem. We follow the ideas in the proofs of similar results for weighted composition operators in [7, 20]. Let  $Q$  be a compact operator on  $F_\alpha^p$ . Since  $\|k_{(w, \alpha)}\|_{(p, \alpha)} = 1$  and  $k_{(w, \alpha)}$  converges to zero uniformly on compact subset of  $\mathbb{C}$  as  $|w| \rightarrow \infty$ , we have

$$\begin{aligned} \|V_{(g, \psi)} - Q\| &\geq \limsup_{|w| \rightarrow \infty} \|V_{(g, \psi)} k_{(w, \alpha)} - Q k_{(w, \alpha)}\|_{(q, \alpha)} \\ &\geq \limsup_{|w| \rightarrow \infty} \|V_{(g, \psi)} k_{(w, \alpha)}\|_{(q, \alpha)} - \|Q k_{(w, \alpha)}\|_{(q, \alpha)} \\ &= \limsup_{|w| \rightarrow \infty} \|V_{(g, \psi)} k_{(w, \alpha)}\|_{(q, \alpha)} \\ &\simeq \left( \limsup_{|w| \rightarrow \infty} B_{(\psi, \alpha)}(|g|^q) \right)^{1/q}, \end{aligned}$$

where the first equality is due to compactness of  $Q$ . To prove the upper inequality, we follow the arguments in the proof of Theorem 1. For each unit norm  $f$  in  $F_\alpha^p$ , we get

$$\begin{aligned} \|V_{(g, \psi)} R_n f\|_{(q, \alpha)}^q &\simeq \int_{\mathbb{C}} |R_n f(z)|^q d\mu_{(q, \alpha)}(z) \\ &\lesssim \int_{\mathbb{C}} e^{\frac{q\alpha}{2}|z|^2} d\mu_{(q, \alpha)}(z) \int_{\mathbb{C}} \chi_{D(z, 1)}(w) |R_n f(w)|^q e^{\frac{p\alpha}{2}|w|^2} dm(w) \\ &\lesssim \int_{\mathbb{C}} |R_n f(w)|^q e^{-\frac{q\alpha}{2}|w|^2} B_{(\psi, \alpha)}(|g|^q)(w) dm(w) \\ &= \left( \int_{\mathbb{C} \setminus D(0, r)} + \int_{D(0, r)} \right) |R_n f(w)|^q e^{-\frac{q\alpha}{2}|w|^2} B_{(\psi, \alpha)}(|g|^q)(w) dm(w) \\ &= I_{n1} + I_{n2} \end{aligned}$$

where for some fixed  $r > 0$ ,

$$\begin{aligned} I_{n1} &= \int_{\mathbb{C} \setminus D(0, r)} |R_n f(w)|^q e^{-\frac{q\alpha}{2}|w|^2} B_{(\psi, \alpha)}(|g|^q)(w) dm(w) \\ &\lesssim \sup_{\mathbb{C} \setminus D(0, r)} B_{(\psi, \alpha)}(|g|^q)(w) \end{aligned}$$

which follows since  $\sup_n \|R_n\| < \infty$ , and

$$I_{n2} = \int_{D(0, r)} |R_n f(w)|^q e^{-\frac{q\alpha}{2}|w|^2} B_{(\psi, \alpha)}(|g|^q)(w) dm(w).$$

We remain to estimate  $I_{n2}$ . By Lemma 1 in [21], we obtain,

$$I_{n2} \lesssim \sup_{w \in \mathbb{C}} B_{(\psi, \alpha)}(|g|^q)(w) I_{n3} \int_{\mathbb{C}} e^{-\frac{q\alpha}{2}|w|^2} dm(w) \quad (2.7)$$

where

$$I_{n3} = \left( \sum_{k=n}^{\infty} \frac{|r|^k}{k!} \left( (2s^{-1})^{2^{-1}sk+1} \Gamma(2^{-1}sk + 1) \right)^{1/s} \right)^q$$

with  $s$  the conjugate exponent of  $p$  and  $\Gamma$  is the Gamma function. By Stirling's formula, it holds

$$\frac{|r|^k}{k!} \left( (2s^{-1})^{2^{-1}sk+1} \Gamma(2^{-1}sk + 1) \right)^{1/s} \simeq \frac{|r|^k}{k!} (2s^{-1})^{2^{-1}k} (2^{-1}sk + 1)^{\frac{k}{2} + \frac{1}{s} + \frac{1}{2s}} e^{-k/2}$$

when  $k \rightarrow \infty$ . By ratio test, the series

$$\sum_{k=0}^{\infty} \frac{|r|^k}{k!} (2s^{-1})^{2^{-1}k} (2^{-1}sk + 1)^{\frac{k}{2} + \frac{1}{s} + \frac{1}{2s}} e^{-k/2}$$

converges and hence  $I_{n3}$  goes to zero when  $n \rightarrow \infty$ . By Theorem 1, it follows that  $I_{n2} \rightarrow 0$  as  $n \rightarrow \infty$ . Therefore

$$\lim_{n \rightarrow \infty} \sup_{\|f\|_{(p, \alpha)} \leq 1} \|V_{(g, \psi)} R_n f\|_{(q, \alpha)}^q \lesssim \sup_{\mathbb{C} \setminus D(0, r)} B_{(\psi, \alpha)}(|g|^q)(w).$$

By Lemma 3 we get

$$\|V_{(g, \psi)}\|_e^q \lesssim \lim_{r \rightarrow \infty} \sup_{\mathbb{C} \setminus D(0, r)} B_{(\psi, \alpha)}(|g|^q)(w) \simeq \lim_{|w| \rightarrow \infty} \sup B_{(\psi, \alpha)}(|g|^q)(w)$$

and completes the proof.

*Proof of Theorem 4.* The crucial step in proving the theorem is to introduce a Teoplitz operator on  $F_{\alpha}^2$ . Let  $\mu$  be a finite positive Borel measure on  $\mathbb{C}$  satisfying the admissibility condition

$$\int_{\mathbb{C}} |K_{(w, \alpha)}(z)|^2 e^{-\alpha|w|^2} d\mu(w) < \infty \quad (2.8)$$

for all  $z \in \mathbb{C}$ . Then we define a Teoplitz operator by

$$T_{\mu} f(z) = \int_{\mathbb{C}} K_{(w, \alpha)}(z) f(w) e^{-\alpha|w|^2} d\mu(w)$$

for each  $z \in \mathbb{C}$ . Since the kernel functions are dense in  $F_{\alpha}^2$ , it follow by the admissibility condition and Hölder's inequality that  $T_{\mu}$  is well-defined. We observe that by Lemma 1, the inner product

$$\langle f, h \rangle = f(0) \overline{h(0)} + \int_{\mathbb{C}} f'(z) \overline{h'(z)} \frac{e^{-\alpha|z|^2}}{(1 + |z|)^2} dm(z) \quad (2.9)$$

defines a norm which is equivalent to the usual norm on  $F_{\alpha}^2$ . We prefer to use this norm since this alternative approach has the advantage that it permits us to associate product of Volterra type integral and composition operators with Teoplitz operators easily. In

deed, if  $V_{(g,\psi)}$  is a bounded operator on  $F_\alpha^2$ , then we claim that  $V_{(g,\psi)}^* V_{(g,\psi)} = T_\mu$  where  $T_\mu$  is the Toeplitz operator induced by the measure  $d\mu = \phi \circ \psi^{-1}$  where

$$\phi(z) = |g'(z)|^2 (1 + |z|)^{-2} e^{-\alpha|z|^2} dm(z).$$

To show the claim, we consider a function  $f$  in  $F_\alpha^2$  and compute

$$\begin{aligned} V_{(g,\psi)}^* V_{(g,\psi)}(f)(z) &= \langle V_{(g,\psi)}^* V_{(g,\psi)} f, K_{(z,\alpha)} \rangle = \langle V_{(g,\psi)} f, V_{(g,\psi)} K_{(z,\alpha)} \rangle \\ &= \int_{\mathbb{C}} f(\psi(w)) \overline{K_{(z,\alpha)}(\psi(w))} |g'(w)|^2 (1 + |w|)^{-2} e^{-\alpha|w|^2} dm(w) \\ &= \int_{\mathbb{C}} f(\psi(w)) K_{(\psi(w),\alpha)}(z) |g'(w)|^2 (1 + |w|)^{-2} e^{-\alpha|w|^2} dm(w) \\ &= \int_{\mathbb{C}} f(\eta) K_{(\eta,\alpha)}(z) e^{-\alpha|\eta|^2} d\mu(\eta) = T_\mu(f)(z), \end{aligned}$$

follows from change of variables. This shows that  $T_\mu = V_{(g,\psi)}^* V_{(g,\psi)}$ . For such particular measure  $\mu$ , the admissibility condition (2.8) holds whenever  $V_{(g,\psi)}$  is bounded on  $F_\alpha^2$ . Denote the associated Berezin symbol  $\tilde{\mu}$  of  $\mu$  by

$$\tilde{\mu}(z) = \langle T_\mu k_{(z,\alpha)}, k_{(z,\alpha)} \rangle, \quad z \in \mathbb{C}.$$

Then an important result from [12] ensures that the Toeplitz operator  $T_\mu$  belongs  $S_p$  if and only if  $\tilde{\mu}$  belongs to  $L^p(\mathbb{C}, dm)$  for each  $p > 0$ . On the other hand,  $V_{(g,\psi)}$  belongs to  $S_p$  if and only if  $V_{(g,\psi)}^* V_{(g,\psi)}$  belongs to  $S_{p/2}$  (see, [23]), and this holds if and only if  $\tilde{\mu}(z) = \|V_{(g,\psi)} k_{(z,\alpha)}\|_{(2,\alpha)}^2$  belongs to  $L^{p/2}(\mathbb{C}, dm)$ . It is easily seen that

$$\|V_{(g,\psi)} k_{(z,\alpha)}\|_{(2,\alpha)}^2 \simeq B_{(\psi,\alpha)}(|g|^2)(z)$$

and completes the proof.

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